

DISPERSION OF STRESS WAVES IN ONE-DIMENSIONAL SEMI-INFINITE, WEAKLY COUPLED LAYERED SYSTEMS

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Abstract—An analytical study of the dispersion of transient stress waves in the first layer of a weakly coupled semi-infinite bi-layered system is carried out. The analysis is based on approximate Fourier transform inversions, and makes use of the fact that the weakly coupled system possesses small propagation zones (PZs) in the frequency domain. Low- and high-frequency asymptotic approximations to the transient waves are computed, taking into account frequency components of the transfer function in the first and second PZs, respectively. The derived analytic expressions are superpositions of non-oscillating terms and convolution integrals with decaying oscillatory kernels. It is shown that, depending on the frequency and the amplitude of the convolution kernels, the dispersed waves overshoot or undershoot the applied impulsive excitation. This result is of significant practical importance in the design of layered systems as stress attenuators. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The propagation of elastic waves in layered media has been studied extensively in the literature (Sun *et al.*, 1968; Delph *et al.*, 1978, 1979a, b; Herrmann and Hemami, 1982; Tygel and Hubral, 1987; Rousseau, 1989). Among other analytical techniques, Floquet theory was employed to study the properties of the dispersion surfaces of propagating SH-, P- and SV-waves, and to compute the boundaries between propagation and attenuation zones (PZs and AZs) in frequency-wavenumber space. Computing transient waves in bounded or unbounded layered media is a challenging task (Tygel and Hubral, 1987; Weaver *et al.*, 1993). Techniques based on integral transforms and transfer matrices suffer from numerical instabilities arising from evanescent waves (exponential dichotomy). Kundu and Mal (1985) and Mal (1988) used a pole removal method to eliminate this numerical instability, and to numerically compute the transient responses of isotropic and anisotropic layered systems excited by harmonic or impulsive point loads. Additional difficulties are encountered in the inversions of the integral transforms to obtain the transient waves (Weaver *et al.*, 1993); these inversions can be performed numerically (Lih and Mal, 1992), using generalized Ray theory (Pao and Gajewski, 1977), analytically/numerically at special limiting regions of the frequency-wavenumber space (Miklowitz, 1962, 1983), or by expressing the transient response as a sum of inverse integral transforms related to individual branches of the dispersion surface (Weaver and Pao, 1982). Tygel and Hubral (1987) analyzed transient waves in layered media by explicitly computing Green functions for point- or line-sources, and expressing the solutions in terms of integrals with finite integration limits.

In additional works dispersion of transient pulses in one-dimensional layered systems was analyzed. Approximate methods relying on small inhomogeneities in the layered medium were developed, based on WKB solutions (Hassab, 1976) or Bremmer series (Mendel,

1978); these simplified approximate theories, however, are not valid when the inhomogeneities in the system are strong. Anfinson (1967) investigated the problem of optimum design of layers for maximizing or minimizing the amplitude of the first transmitted stress wave; his formulation relied on numerically optimizing solutions of sets of difference equations. In Achenbach *et al.* (1968) the transmission of primary pulses through one-dimensional layered systems was analytically and experimentally studied, and it was shown that laminated systems consisting of layers with very different mechanical impedances are prone to first-interface bond failure under compressive excitations of very short durations. Similar conclusions were derived by Vakakis *et al.* (1994) who asymptotically analyzed the structure of propagation and attenuation zones (PZs and AZs) of bi-layered one-dimensional layered media at the limit of large impedance differences between layers. In that work parameters controlling the width of the PZs were identified, analytic approximations for the boundaries of the PZs were derived, and the effects of localized disorders on the propagation of stress waves were analytically and numerically studied. Carcione *et al.* (1991) investigated numerically the validity of the long-wave approximation theory (Santosa and Symes, 1991) in one- and two-dimensional semi-infinite layered media; they confirmed the non-dispersive transversely isotropic behavior of these media at long wavelengths, and their scattering behavior at short wavelengths. Norris (1993) compared two asymptotic theories for one-dimensional wave propagation in in-homogeneous media: the long-wavelength approximation (Santosa and Symes, 1991) and the O'Doherty-Anstey (1971) theory for wave propagation in finely layered media with small reflectivity. The equivalence of these two theories in their common range of validity was analytically established. Tenenbaum and Zindeluk (1992) derived an exact algebraic scheme for computing transient waves in one-dimensional layered media excited by arbitrary input pulses; their scheme was based on the method of characteristics, and, due to the amount of algebraic manipulations required for its implementation, it was used iteratively.

In the present work the dispersion of stress waves in a weakly coupled, one-dimensional semi-infinite bi-layered medium is analytically studied. Denoting by τ the ratio of mechanical impedances of the two layers, it was shown (Vakakis *et al.*, 1994) that for large τ , $\tau = O(1/\varepsilon)$, $0 < \varepsilon \ll 1$ (i.e., when the medium is composed of weakly coupled, stiff layers), the bi-layered system possesses degenerate and non-degenerate PZs with widths of $O(\varepsilon^{1/2})$ and $O(\varepsilon)$, respectively. Use of this result is made herein to asymptotically study the dispersion of transient stress waves in the limit of small (but non-zero) ε . The analysis is developed for general forms of transient excitations, and relies on asymptotic evaluations of inverse Fourier transforms. This task is demanding since separate asymptotic approximations of the Fourier integrands need to be developed at different frequency ranges representing propagation or attenuation zones of the layered medium; in addition, a careful justification of the asymptotic operations must be performed, in order to show that the derived analytical expressions are the dominant terms of the stress wave, and to estimate the orders of magnitude of higher order (neglected terms). We will perform the aforementioned asymptotic Fourier inversions only in the limit of weak coupling between layers, since only then can the assumptions of the asymptotic theory be justified.

To our knowledge, *analytical* studies of *transient dispersions* of stress waves in layered media are not very common in the literature, due to the difficulties associated with the analytical inversions of the associated integral transforms, or, for example, with the very large number of rays required in ray theory. Previous studies of dispersions of transient pulses in linear one-dimensional spring-mass chains were carried out by Weinstock (1970), Nayfeh and Rice (1972), Lee (1972), and Wang and Lee (1973), who explicitly analyzed the transient responses of the chains due to general excitations; these studies were performed by employing normal mode decompositions, or by analytically inverting Laplace or Fourier transforms.

2. MOTIVATION FOR THIS STUDY

The semi-infinite, monocoupled undamped layered system considered herein is depicted in Fig. 1. It consists of identical periodic sets, composed of two homogeneous,

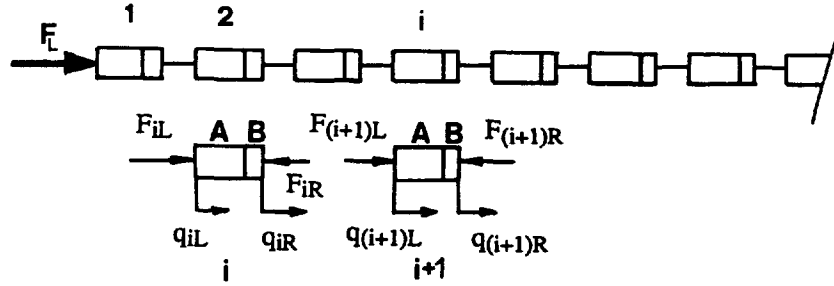


Fig. 1. The semi-infinite, mono-coupled, bi-layered system.

linearly elastic layers, labeled 'A' and 'B', respectively. The free and forced dynamics of the corresponding infinite and finite layered systems were studied by Vakakis *et al.* (1994) by introducing a nondimensional frequency ω and structural parameters τ and ν , defined by the relations:

$$\omega = \Omega(l_A/c_A), \quad \tau = (A_A E_A/c_A)/(A_B E_B/c_B), \quad \nu = (l_B/c_B)/(l_A/c_A) \quad (1)$$

In (1), Ω is the dimensional frequency in (rad/sec), l_A, l_B the layer thicknesses, A_A, A_B the layer cross sections, E_A, E_B the moduli of elasticity and c_A, c_B the speeds of sound in the two layers. Parameter τ represents the ratio of mechanical impedances of the two layers, whereas ν represents the ratio of times of travel of waves propagating at phase velocity through each layer.

Considering the i -th periodic element of the infinite system in Fig. 1, and assuming harmonic oscillations, the displacements and internal forces at the right and left boundaries of this element are related by the transfer matrix:

$$\begin{Bmatrix} q_{(i+1)L} \\ F_{(i+1)L} \end{Bmatrix} = \begin{bmatrix} t_{LL} & t_{LR} \\ t_{RL} & t_{RR} \end{bmatrix} \begin{Bmatrix} q_{iL} \\ F_{iL} \end{Bmatrix} \quad (2a)$$

(cf. Fig. 1), where q_{iR}, q_{iL} denotes internal displacements and F_{iR}, F_{iL} internal stresses. The components $t_{mn}, m, n = L, R$, satisfy the relation $t_{LL}t_{RR} - t_{LR}t_{RL} = 1$, and are given by (Vakakis *et al.*, 1994):

$$\begin{aligned} t_{LL} &= \cos \nu \omega \cos \omega - \tau \sin \nu \omega \sin \omega, & t_{RR} &= \cos \nu \omega \cos \omega - (1/\tau) \sin \nu \omega \sin \omega \\ t_{LR} &= -(A_A E_A/l_A)^{-1} \omega^{-1} [\cos \nu \omega \sin \omega + \tau \sin \nu \omega \cos \omega] \\ t_{RL} &= (A_A E_A/l_A) \omega [\cos \nu \omega \sin \omega + (1/\tau) \sin \nu \omega \cos \omega] \end{aligned} \quad (2b)$$

The propagation and attenuation zones (PZs and AZs) of the layered medium are computed by relating displacements and forces at the right and left boundaries of a periodic element by the expressions,

$$q_{iR} = q_{(i+1)L} = e^\mu q_{iL}, \quad F_{iR} = -F_{(i+1)L} = -e^\mu F_{iL} \quad (3)$$

and evaluating the *propagation constant* μ by substituting (3) into (2a). Wave propagation in the infinite layered system can occur only when μ is a purely imaginary scalar. Moreover, it can be shown that if μ is a solution, so is $-\mu$; this positive-negative pair of propagation constants corresponds to two *characteristic waves* propagating in opposite directions in the infinite system. In Vakakis *et al.* (1994) it was shown that when τ is small or large (of $O(\varepsilon)$ or $O(1/\varepsilon)$, where $0 < \varepsilon \ll 1$), the infinite system possesses small propagation zones (PZs) in the frequency domain. The bounding frequencies separating PZs and AZs of this system are solutions of the equation,

$$\cosh \mu = \pm 1 \tag{4a}$$

where (\pm) signs in represent limiting cases of standing waves (Mead, 1975). Taking into account (2) and (3) relation (4a) can be expressed as,

$$[-(1 + \varepsilon^2 \tau^{*2}) \sin \omega \sin v\omega + 2\varepsilon\tau^* \cos \omega \cos v\omega] = \pm 2\varepsilon\tau^* \tag{4b}$$

where the notation $\tau = \tau^*/\varepsilon$, $\tau^* = O(1)$, was introduced to denote large magnitude of parameter τ .

By solving (4b) two types of PZs can be identified (Vakakis *et al.*, 1994), namely, *degenerate* PZs, defined by bounding frequencies of the form :

$$\omega_{bp}^{(1)\pm} = p\pi \pm \varepsilon^{1/2} \{[-2\tau^{*-1/2} v^{-1/2}][1 - ((1 + v^2)\varepsilon/2\tau^*v)]\} + O(\varepsilon^{5/2}) \quad (\sin vp\pi = 0) \tag{5a}$$

$$\omega_{bp}^{(2)\pm} = p\pi/v \pm \varepsilon^{1/2} \{[-2\tau^{*-1/2} v^{-1/2}][1 - ((1 + v^2)\varepsilon/2\tau^*v)]\} + O(\varepsilon^{5/2}) \quad (\sin p\pi/v = 0) \tag{5b}$$

$p = 0, 1, 2, \dots$, and *non-degenerate* PZs with bounding frequencies :

$$\omega_{bk}^{(1)\pm} = k\pi + \varepsilon \{[-2((-1)^{k+1} \cos vk\pi \pm 1)]/(-1)^k \tau^* \sin vk\pi\} + O(\varepsilon^2), \quad (\sin vk\pi \neq 0) \tag{6a}$$

$$\omega_{bk}^{(2)\pm} = k\pi/v + \varepsilon \{[-2((-1)^{k+1} \cos k\pi/v \pm 1)]/(-1)^k v\tau^* \sin k\pi/v\} + O(\varepsilon^2), \quad (\sin k\pi/v \neq 0) \tag{6b}$$

where $k = 1, 2, \dots$, $\tau = \tau^*/\varepsilon$, and all variables other than ε are assumed to be $O(1)$ quantities.

From expressions (5) and (6) one notes that when τ is of $O(1/\varepsilon)$, the corresponding widths of the PZs of the infinite system are of $O(\varepsilon^{1/2})$ (degenerate PZs) or $O(\varepsilon)$ (non-degenerate PZs), and, thus, small. This is the case when layer *A* is composed of stiff material, such as, ceramic, and layer *B* of soft material, for example in a polymer. Indeed, as $\tau \rightarrow \infty$ the widths of the PZs tend to zero. An additional observation regarding the bounding frequencies (5) and (6) is that parameter v affects the positions of the PZs in the frequency domain. To demonstrate the effect of v on the structure of the PZs, consider two bi-layered systems of infinite extent, denoted from now on as systems ‘I’ and ‘II’. The structural parameters of the two systems are listed below.

System I		System II	
$l_A = 0.495$ in	$l_B = 0.005$ in	$l_A = 0.450$ in	$l_B = 0.050$ in
$E_A = 4.64 \times 10^7$ lb/in ²	$E_B = 1 \times 10^4$ lb/in ²	$E_A = 4.64 \times 10^7$ lb/in ²	$E_B = 1 \times 10^4$ lb/in ²
$A_A = 1$ in ²	$A_B = 1$ in ²	$A_A = 1$ in ²	$A_B = 1$ in ²
$\rho_A = 3.04 \times 10^{-4}$ lb/in ³	$\rho_B = 1 \times 10^{-4}$ lb/in ³	$\rho_A = 3.04 \times 10^{-4}$ lb/in ³	$\rho_B = 1 \times 10^{-4}$ lb/in ³
$c_A = 3.91 \times 10^5$ in/sec	$c_B = 1 \times 10^4$ in/sec	$c_A = 3.91 \times 10^5$ in/sec	$c_B = 1 \times 10^4$ in/sec
$\tau^* = 1.18, v = 0.394, \varepsilon = 10^{-2}$		$\tau^* = 1.18, v = 4.341, \varepsilon = 10^{-2}$	

Note that the only difference between systems I and II is the lengths of their ‘stiff’ and ‘soft’ layers, and that both systems possess the same (large) value of τ , but different values of v . In Fig. 2 the real and imaginary parts of the propagation constants of the two infinite systems are depicted vs non-dimensional frequency ω , and the location of the PZs is indicated. Both systems I and II possess low-frequency degenerate PZs centered at $\omega = 0$ (labeled as ‘1PZs’) of widths approximately equal to 2.877×10^{-1} and 0.881×10^{-1} , respectively (note that both 1PZs are of $O(\varepsilon^{1/2}) = O(10^{-1})$). The first non-degenerate PZ for system I occurs close to $\omega = \pi$ and is of $O(\varepsilon) = O(10^{-2})$. By contrast, for $\omega < \pi$ system II possesses numerous non-degenerate PZs, each of $O(10^{-2})$ and centered at $\omega = \pi/v$ (‘2PZ’), $\omega = 2\pi/v$ (‘3PZ’), or at higher frequencies.

It is now shown that the different structure of the PZs of the *infinite* systems I and II affects significantly the dispersion of propagating stress waves at the leading layers of the corresponding *semi-infinite* systems. To this end, a trapezoidal pulse of 25 μ s duration and

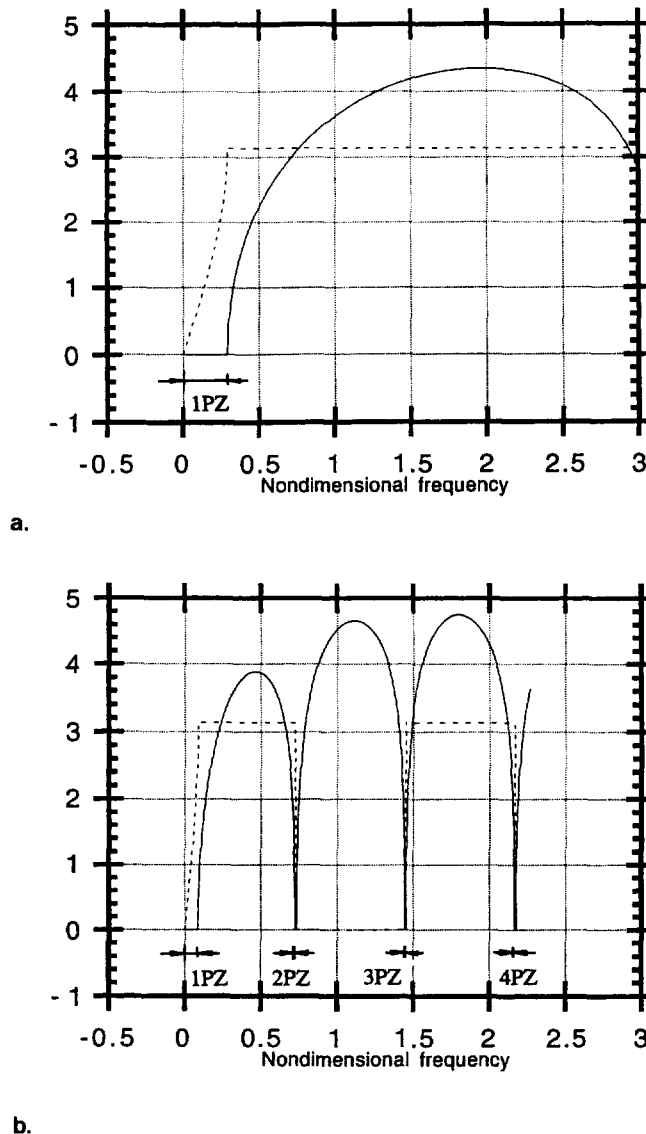


Fig. 2. Real (—) and imaginary (---) parts of the propagation constant μ : (a) system I, and (b) system II.

amplitude 10^5 lb/in² was applied to the free boundaries of the semi-infinite systems I and II, and the generated compressional stress waves were computed at various positions of the first layer A .

The computations were carried out by performing numerical inverse-Fast Fourier Transforms (IFFTs) of the Fourier-transformed internal stresses in the first layer of the semi-infinite layered system; the analytical expressions of the transformed stresses are given in the next section. Although this procedure is conceptually simple, a fine sampling of the frequency response is required for accurate computational results; moreover, as a remedy of numerical instabilities associated with exponential dichotomy (large and small real eigenvalues of the transfer matrices inside AZs) a loss factor equal to $\eta = 2.5 \times 10^{-2}$ was assumed for all layers, leading to complex-valued moduli of elasticity $E = E_0(1 + j\eta)$. A Nyquist frequency of $f_N = 4 \times 10^5$ Hz and a sampling frequency of $\Delta f = 24.414$ Hz were employed in the spectral computations, leading to a total number of $2^{15} = 32768$ sample points. The accuracy of such spectral computations for finite layered media was checked by an independent set of calculations based on modal superposition (El-Raheb, 1993; Vakakis *et al.*, 1994).

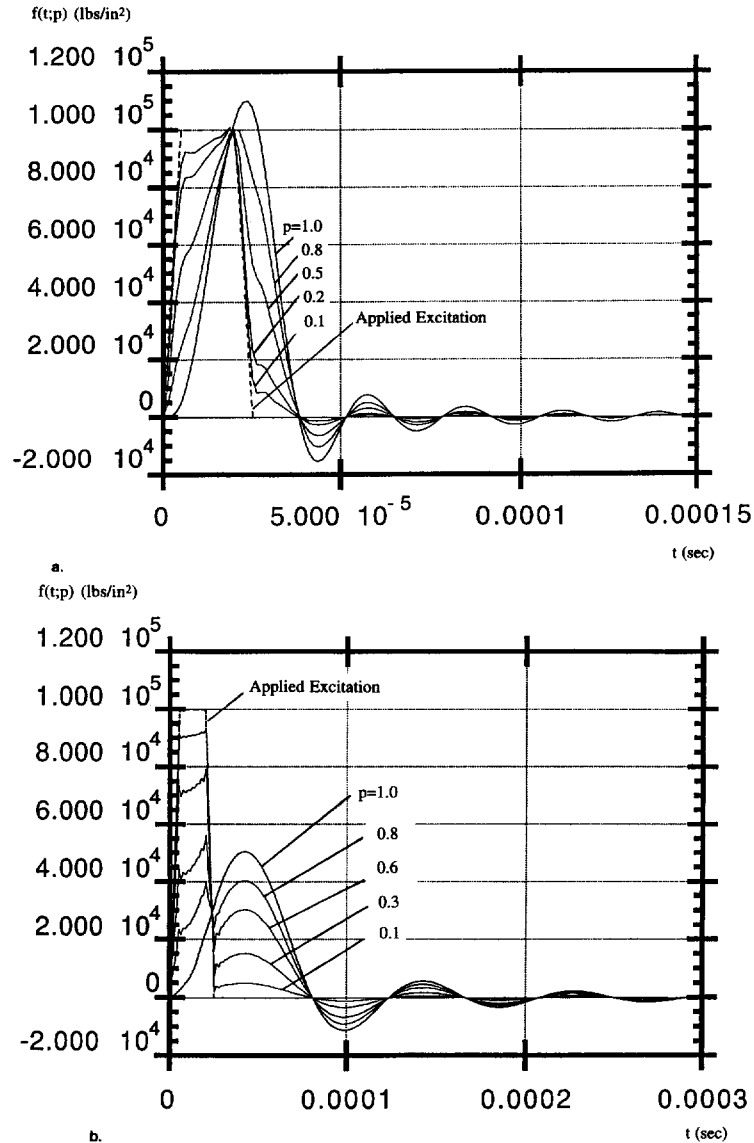


Fig. 3. Numerical computation of the dispersed stress wave in the first 'stiff' layer : (a) system I, and (b) system II. The applied load is of trapezoidal shape.

In Fig. 3a the compressional stress wave is depicted at stations $p = s/l_A = 0.1, 0.2, 0.5, 0.8, 1.0$ of the first layer A of system I. The corresponding stress wave for system II is presented in Fig. 3b at positions $p = s/l_A = 0.1, 0.3, 0.6, 0.8, 1.0$. Comparing the dispersion of the trapezoidal stress pulse in the two systems one concludes the following.

- In system I the maximum amplitude of the dispersed stress pulse is greater than the amplitude of the applied force; that is, the dispersed wave *overshoots* the applied impulsive load.
- By contrast, in system II the dispersed wave drastically *undershoots* the applied force.
- In both systems there exist high-frequency 'tails' accompanying the dispersed stress pulses. The 'tail' of the dispersed wave in system I is of higher frequency than that of system II.
- Sufficiently close to the free surface of system II there exist additional low-amplitude, high-frequency oscillations superimposed to the main stress pulse; such high-frequency oscillations are not observed in the dispersed pulse of system I.

From the above it follows that the semi-infinite systems I and II possess radically different dispersion properties: system I amplifies through dispersion the applied stress pulse, whereas system II drastically attenuates it. The numerical results depicted in Fig. 3 provide the primary motivation for performing the asymptotic analysis of the following sections. The overall goal of this analysis is a better understanding of the dispersion of transient waves in layered media for the purpose of employing such media as shock isolators against external impulsive loads.

3. DISPERSION OF THE STRESS WAVE—LOW FREQUENCY APPROXIMATION

Analytical expressions for the Fourier-transformed internal stress in the first layer of the semi-infinite system of Fig. 1 can be derived, either by employing the method of characteristic receptances first introduced by Mead (1975), or by considering (2a) and (3) and imposing the causality condition that there is only one (right-going) characteristic wave propagating through the system. Denoting by $F_L(\omega)$ the Fourier transform of the applied stress $f_L(t)$, it can then be shown that the Fourier-transformed internal stress $F(\omega; p)$ at position $p = s/l_A$ in layer A (the 'stiff' layer) of the first periodic set of the semi-infinite system is given by:

$$F(\omega; p) = \left[\cos(p\omega) + \sin(p\omega) \frac{(1 - \tau^2)(2\tau)^{-1} \sin(v\omega) \sin \omega - \sinh \mu}{\cos(v\omega) \sin \omega + (1/\tau) \sin(v\omega) \cos \omega} \right] F_L(\omega) \equiv R(\omega; p) F_L(\omega) \quad (7)$$

where ω is the previously defined non-dimensional frequency, and $R(\omega; p) = F(\omega; p)/F_L(\omega)$ the transfer function. The propagation constant μ is computed by the following expression (Vakakis *et al.*, 1994):

$$\cosh \mu = (2\tau)^{-1} [-(1 + \tau^2) \sin \omega \sin v\omega + 2\tau \cos \omega \cos v\omega] \quad (8)$$

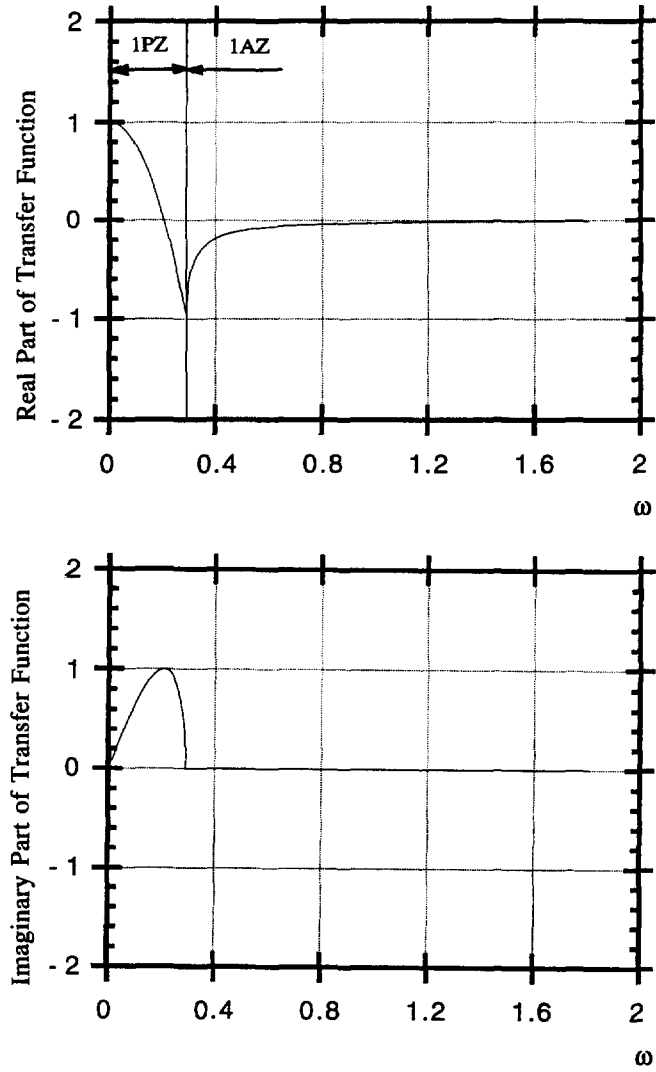
There exist two branches of μ , corresponding to positive- and negative-going waves in the layered system of infinite extent. As discussed previously, due to the causality only the positive branch of μ should be considered in computing waves propagating in the semi-infinite system. Expressions similar to (7) hold for the transformed internal stresses in other layers of the system, but since the present work focuses on the initial dispersion of the applied transient stress pulse, only waves in the first layer are considered.

Examining expressions (7) and (8), at $p = 0$ (traction-free surface) it is satisfied that $F(\omega; p) = F_L(\omega)$, whereas at $p = 1$ (first interface between layers) the applied pulse is fully dispersed. In Fig. 4 the transfer functions $R(\omega; 1) = F(\omega; 1)/F_L(\omega)$ at position $p = 1$ of systems I and II are depicted. It is noted that the transfer functions attain finite amplitudes at small neighborhoods of the PZs of the corresponding infinite systems, and nearly vanish elsewhere. This feature of the transfer functions is employed in the following analysis to construct closed-form asymptotic approximations of integrals associated with the Fourier inversion of expression (7).

The transient stress wave in the leading layer of the semi-infinite systems is computed by inverting the set (7), (8). The following convolution integral is then derived:

$$f(t; p) = (2\pi)^{-1} (c_A/l_A) \int_{-\infty}^{+\infty} R(\omega; p) F_L(\omega) \exp(j(c_A/l_A)\omega t) d\omega = \int_{-\infty}^{+\infty} r(t - \tau; p) f_L(\tau) d\tau \quad (9a)$$

where the kernel $r(t; p)$ is the inverse Fourier transform of the transfer function $R(\omega; p)$:

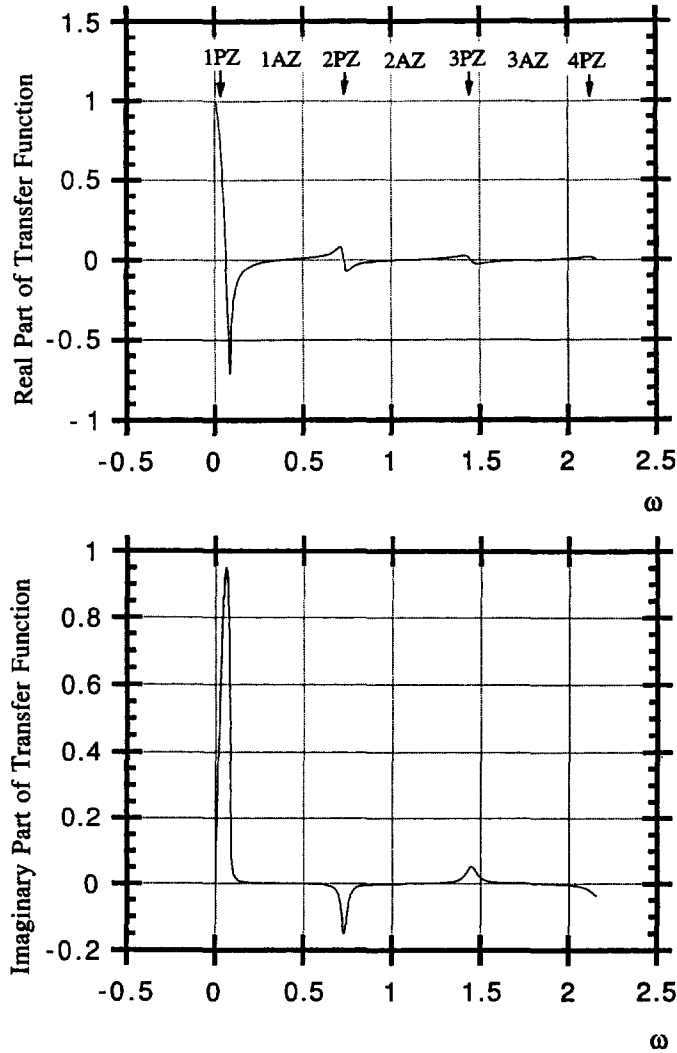


a.

Fig. 4. Transfer functions $F(\omega; 1)/F_L(\omega)$ at the first layer interface ($p = 1$) of the semi-infinite systems: (a) system I, and (b) system II. The positions of the PZs of the corresponding infinite systems are also depicted.

$$\begin{aligned}
 r(t; p) &= (2\pi)^{-1} (c_A/l_A) \int_{-\infty}^{+\infty} R(\omega; p) \exp(j(c_A/l_A)\omega t) d\omega \\
 &= (2\pi)^{-1} (c_A/l_A) \int_{-\infty}^{+\infty} \left[\cos(p\omega) + \sin(p\omega) \frac{(1-\tau^2)(2\tau)^{-1} \sin(v\omega) \sin \omega - \sinh \mu}{\cos(v\omega) \sin \omega + (1/\tau) \sin(v\omega) \cos \omega} \right] \\
 &\quad \times \exp(j(c_A/l_A)\omega t) d\omega \quad (9b)
 \end{aligned}$$

In (9a), $f(t; p)$ denotes the transient stress wave at position p of the first layer, and $j = (-1)^{1/2}$. In the following analysis an asymptotic approximation to the kernel $r(t; p)$ of the convolution integral is derived. To a first approximation, only low-frequency components of the transfer function $R(\omega; p)$ will be taken into account, and $r(t; p)$ will be approximated as follows:



b.

Fig. 4. Continued.

$$r(t; p) \approx r^{(LF)}(t; p) = (2\pi)^{-1} (c_A/l_A) \left\{ \int_{-\infty}^{-\omega_{b_0}^{(1,2)}} R^{(LF)}(\omega; p) \exp(j(c_A/l_A)\omega t) d\omega + \int_{-\omega_{b_0}^{(1,2)}}^{+\omega_{b_0}^{(1,2)}} R^{(LF)}(\omega; p) \exp(j(c_A/l_A)\omega t) d\omega + \int_{+\omega_{b_0}^{(1,2)}}^{+\infty} R^{(LF)}(\omega; p) \exp(j(c_A/l_A)\omega t) d\omega \right\} \quad (10)$$

where $r^{(LF)}(t; p)$ denotes the low-frequency approximation of $r(t; p)$, $\pm \omega_{b_0}^{(1,2)}$ are the frequency boundaries of 1PZ (cf. eqn (5a, b), with $p = 0$ and negative frequencies allowed), and $R^{(LF)}(\omega; p)$ is the low frequency approximation of the transfer function; the quantity $R^{(LF)}(\omega; p)$ is analytically evaluated below. Motivated by the fact that for $\tau = O(1/\epsilon)$ the 1PZs of systems I and II are degenerate, and of $O(\epsilon^{1/2})$, the frequency ω is rescaled as $\omega = \epsilon^{1/2}x$, where x is an $O(1)$ frequency parameter. Rescaling ω in the expression of the propagation constant, (8), and selecting only the positive branch μ , one obtains the following low-frequency asymptotic approximation for $\sinh \mu$ at frequencies inside 1PZ and 1AZ:

$$\sinh \mu^{(LF)} = j(v\tau^*)^{1/2} x \left[1 - \frac{v\tau^* x^2}{4} \right]^{1/2} + O(\varepsilon),$$

$$\text{for } \omega_{b0}^{(1,2)} = 2\varepsilon^{1/2} (v\tau^*)^{-1/2} + O(\varepsilon^{3/2}) \geq |\omega| = \varepsilon^{1/2} |x| \quad (1PZ)$$

$$\sinh \mu^{(LF)} = -(v\tau^*)^{1/2} x \left[\frac{v\tau^* x^2}{4} - 1 \right]^{1/2} + O(\varepsilon),$$

$$\text{for } \omega_{b0}^{(1,2)} = 2\varepsilon^{1/2} (v\tau^*)^{-1/2} + O(\varepsilon^{3/2}) < |\omega| = \varepsilon^{1/2} |x| \quad (1AZ) \quad (11)$$

The above expressions hold at the limit of large τ , when $\tau = \tau^*/\varepsilon$. Rescaling ω in the expression of $R(\omega; p)$ (eqn (7)), expanding in terms of ε , taking into account (11), and retaining only terms of leading-order, one obtains the following low-frequency asymptotic approximation for $R(\omega; p)$:

$$\begin{aligned} R^{(LF)}(\omega; p) &= R^{(LF)}(\varepsilon^{1/2} x; p) \\ &= \left[1 - \frac{pv\tau^* x^2}{2} \right] - j \left[(v\tau^*)^{1/2} px \left[1 - \frac{v\tau^* x^2}{4} \right]^{1/2} \right] + O(\varepsilon) \quad (1PZ) \\ R^{(LF)}(\omega x; p) &= R^{(LF)}(\varepsilon^{1/2}; p) \\ &= \left[1 - \frac{pv\tau^* x^2}{2} \right] + (v\tau^*)^{1/2} px \left[\frac{v\tau^* x^2}{4} - 1 \right]^{1/2} + O(\varepsilon) \quad (1AZ) \end{aligned} \quad (12)$$

Expressions (12) are asymptotically valid only in small neighborhoods of 1PZ. Since the computation of the kernel $r^{(LF)}(t; p)$ requires integration of integrands defined over the entire frequency range $-\infty < \omega < +\infty$ (cf. eqn (7)), it is necessary to extend the local approximations (12) over the entire frequency range. This is performed by neglecting terms of $O(\varepsilon)$ or of higher order, and expressing the quantities x and τ^* in terms of the original variables ω and τ . The following approximate expression for the low-frequency approximation $R^{(LF)}(\omega; p)$ is then obtained, which is valid over the entire frequency range:

$$\begin{aligned} R^{(LF)}(\omega; p) &= \left[1 - \frac{pv\tau\omega^2}{2} \right] - j \left[(v\tau)^{1/2} p\omega \left[1 - \frac{v\tau\omega^2}{4} \right]^{1/2} \right], \quad \omega_{b0}^{(1,2)} \geq |\omega| \\ R^{(LF)}(\omega; p) &= \left[1 - \frac{pv\tau\omega^2}{2} \right] + \left[(v\tau)^{1/2} p\omega \left[\frac{v\tau\omega^2}{4} - 1 \right]^{1/2} \right], \quad \omega_{b0}^{(1,2)} < |\omega| \end{aligned} \quad (13)$$

Substituting (13) into (7), introducing the change of variable, $v = 2^{-1/2}(v\tau)^{1/2}\omega$, and expanding the complex exponentials of the integrands in terms of trigonometric functions, one obtains the following expression for the low-frequency kernel $r^{(LF)}(t; p)$:

$$\begin{aligned} r(t; p) \approx r^{(LF)}(t; p) &= (2/\pi)(c_A/l_A)(v\tau)^{-1/2} \left\{ \int_0^1 (1-2pv^2) \cos(\alpha vt) dv \right. \\ &\quad \left. + \int_0^1 2pv(1-v^2)^{1/2} \sin(\alpha vt) dv + \int_1^{+\infty} [1-2pv^2 + 2pv(1-v^2)^{1/2}] \cos(\alpha vt) dv \right\} \quad (14) \end{aligned}$$

where $\alpha = 2(c_A/l_A)(v\tau)^{-1/2}$. An intermediate calculation is required before evaluating the integrals (14). It can be shown, that by appropriate definition of the multi-valued function $f(z) = (z^2 - 1)^{1/2}$, $z \in \mathbf{C}$ on the real axis of the complex plane, the following relation holds:

$$\begin{aligned} & \int_{-\infty}^{+\infty} [1 - 2v^2 + 2vf(v)] \exp(-j\alpha vt) \, dv \\ &= 2 \left\{ \int_0^1 (1 - 2v^2) \cos(\alpha vt) \, dv + \int_0^1 2v(1 - v^2)^{1/2} \sin(\alpha vt) \, dv \right. \\ & \quad \left. + \int_1^{+\infty} [1 - 2v^2 + 2v(1 - v^2)^{1/2}] \cos(\alpha vt) \, dv \right\} \quad (15a) \end{aligned}$$

The left-hand side of eqn (15a) can be evaluated by contour integration (Wang and Lee, 1973) as follows:

$$\int_{-\infty}^{+\infty} [1 - 2v^2 + 2vf(v)] \exp(-j\alpha vt) \, dv = (4\pi/\alpha t) J_2(\alpha t) \quad (15b)$$

where $J_2(\cdot)$ is the second-order Bessel function of the first kind. Taking into account the relation (Gradshteyn and Ryzhik, 1980),

$$\int_0^1 2v(1 - v^2)^{1/2} \sin(\alpha vt) \, dv = (\pi/\alpha t) J_2(\alpha t) \quad (15c)$$

and combining (15a-c), the following equality is established:

$$\int_1^{+\infty} [1 - 2v^2 + 2v(1 - v^2)^{1/2}] \cos(\alpha vt) \, dv = (\pi/\alpha t) J_2(\alpha t) - \int_0^1 (1 - 2v^2) \cos(\alpha vt) \, dv \quad (15d)$$

The integral on the right-hand side of the above expression can be easily evaluated in terms of elementary functions, and, thus, the improper integral (15d) is explicitly determined. Taking into account the intermediate analytical results (15a-d), expression (14) is evaluated in closed form as follows:

$$r(t; p) \approx r^{(LF)}(t; p) = (1 - p)\delta(t) + \frac{2p}{t} J_2(\alpha t) \quad (16)$$

where the definition of Dirac's generalized function,

$$\delta(t) = (1/2\pi) \int_{-\infty}^{+\infty} \exp(jut) \, du,$$

was employed.

Hence, the low-frequency kernel $r^{(LF)}(t; p)$ is determined in closed form. Substituting (16) into the convolution integral (9a), performing algebraic manipulations, and taking into account the properties of Dirac's function one obtains the following analytical expression for the transient stress wave:

$$\begin{aligned} f(t; p) &\approx f^{(LF)}(t; p) = \int_{-\infty}^{+\infty} r^{(LF)}(t - \tau; p) f_L(\tau) \, d\tau \\ &= (1 - p)f_L(t) + 2p \int_{-\infty}^{+\infty} \frac{J_2(\alpha(t - \tau))}{(t - \tau)} f_L(\tau) \, d\tau, \quad \alpha = 2(c_A/l_A)(v\tau)^{-1/2} \quad (17) \end{aligned}$$

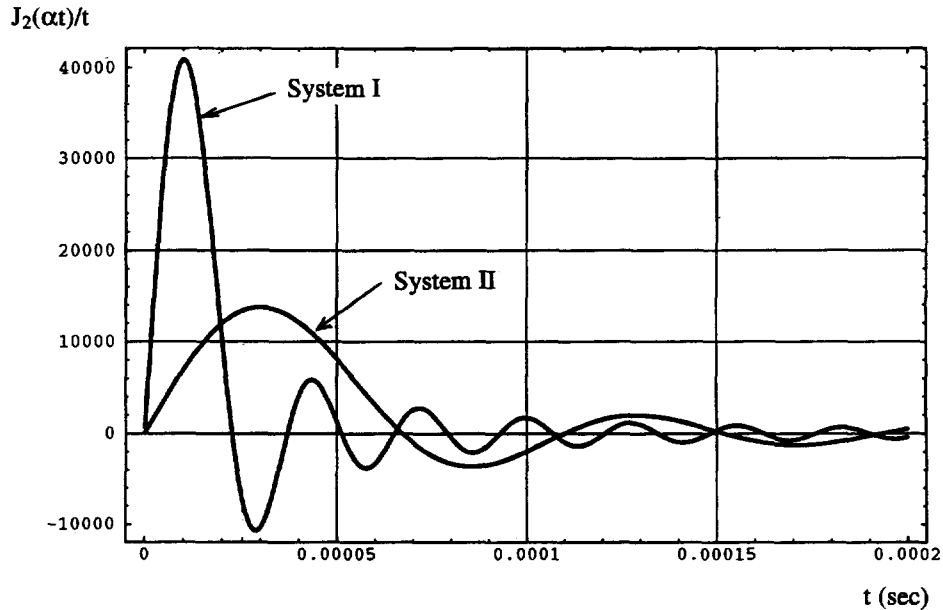


Fig. 5. Oscillatory kernel $J_2(xu)/u$ of the convolution integral (14) for systems I and II.

Expression (17) asymptotically approximates the transient stress wave in the first ‘stiff’ layer of the semi-infinite system in the limit of large values of τ . In deriving (17) only low-frequency components of the transfer function were taken into account, an approximation which is justified when the impedance parameter τ is large, since then, the first degenerate PZ is of $O(\tau^{-1/2})$, and, thus, small; moreover, higher-frequency non-degenerate PZs are of $O(\tau^{-1})$ (i.e., smaller than 1PZ), and provide minimal corrections to the overall transient response.

The analytical expression (17) indicates that the transient wave in the first layer is composed of two terms. The first term is directly proportional to the applied load, does not depend on the structural parameters of the layered system, and diminishes in amplitude as p increases; at $p = 1$, i.e., when the stress pulse is fully dispersed, this term vanishes. The second term in (17) is oscillatory, and is in the form of a convolution of an oscillatory kernel and the applied load. The parameter α in the argument of the Bessel function is proportional to the width of the first PZ of the layered system; for system I, $\alpha = 2.27 \times 10^5$, whereas, for system II, $\alpha = 7.65 \times 10^4$. Hence, there is an order of magnitude difference in the values of α for the two systems, a feature which explains the overshoot or undershoot of the dispersed stress waves of Fig. 3. The kernel $J_2(xu)/u$ of the convolution integral (17) for systems I and II is depicted in Fig. 5; note that there exist large differences in the magnitudes and frequencies of the kernels, caused by the large differences of the corresponding values of α . When the kernels of Fig. 5 are convoluted with the applied load, they produce overshoot (system I) or undershoot (system II) of the dispersed stress wave in the first layer. Moreover, the frequencies of the kernels determine the frequencies of the ‘tails’ of the dispersed pulses. In Fig. 6 the analytical transient waves (17) for systems I and II are depicted for various positions $0 \leq p \leq 1$. These results were obtained by numerically evaluating the convolution integral (17) using Mathematica. Comparing these plots with the numerical results of Fig. 3 good agreement is observed. The analytical formula (17) captures the main (low-frequency) characteristics of the dispersed stress pulse, although, due to the previous simplifying assumptions it does not model the low-amplitude, high frequency modulations of the wave in system II. These modulations are caused by frequency components of the transfer function close to higher PZs of the layered system, which were not taken into account in the low frequency approximation (17). In the next section an approximate asymptotic analysis of the high-frequency modulations of the stress wave in system II is carried out.

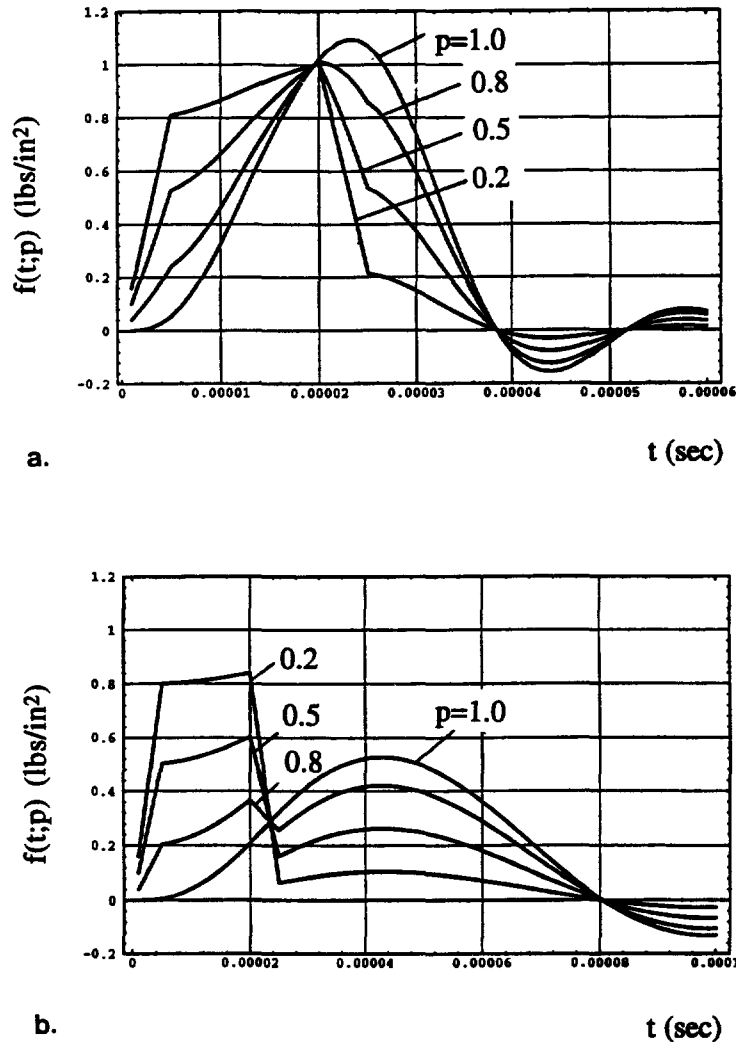


Fig. 6. Analytic approximation of the dispersed stress wave in the first 'stiff' layer: (a) system I, and (b) system II. Results based on formula (14).

4. DISPERSION OF THE STRESS WAVE—HIGH FREQUENCY EFFECTS

As mentioned in the previous sections, the high-frequency modulations appearing in the numerical computations of the stress wave for system II (Fig. 3b) are caused by frequency components of the transfer function lying in vicinities of high-order non-degenerate PZs. The fact that no high-frequency modulations exist in the wave propagating in system I can be easily justified by examining the structure of PZs of this system. Referring to the plots of Fig. 2, 2PZ in system I occurs at relatively high-frequencies ($\omega > 3$); this feature, coupled with the fact that the strongest frequency components of the impulsive load are confined to low-frequencies (in the range $|\omega| < 1$), leads to negligible contributions from high-order PZs in the propagating pulse in system I. By contrast, system II possesses numerous non-degenerate PZs in the range $|\omega| < 3$, giving rise to the high-frequency modulations observed in the numerical results of Fig. 3a. Hence, in the following analysis only system II is considered. Moreover, only frequency components of the transfer function in the vicinity of 2PZ (cf. Fig. 2b) are analyzed; the effects on the response of higher order PZs can be analyzed similarly.

The exact expression of the transient stress wave $f(t;p)$ in the first 'stiff' layer of system II is provided by expressions (9a, b). To account for high-frequency effects due to 2PZ, $f(t;p)$ is approximated as follows:

$$f(t; p) \approx f^{(LF)}(t; p) + f^{(HF)}(t; p) \quad (18)$$

where $f^{(LF)}(t; p)$ is given by (17), and the high-frequency correction $f^{(HF)}(t; p)$ is approximated according to:

$$f^{(HF)}(t; p) = \int_{-\infty}^{+\infty} r^{(HF)}(t-\tau; p) f_L(\tau) d\tau, \quad (19)$$

$$r^{(HF)}(t; p) = (2\pi)^{-1} (c_A/l_A) \int_{-\infty}^{+\infty} R^{(HF)}(\omega; p) \exp(j(c_A/l_A)\omega t) d\omega$$

The high-frequency contribution to the transfer function due to 2PZ, $R^{(HF)}(\omega; p)$, is computed in a way similar to $R^{(LF)}(\omega; p)$, i.e., by asymptotically approximating the transfer function sufficiently close to 2PZ, and extending the resulting local approximations over the entire frequency range $-\infty < \omega < +\infty$. In the vicinity of 2PZ, the frequency ω is parametrized according to $\omega = \pi/v + \varepsilon x$; for large $\tau = \tau^*/\varepsilon$, $\cosh \mu$ is asymptotically approximated by expanding the exact expression (8) in terms of ε , and retaining only leading-order terms:

$$\cosh \mu = -\cos(\pi/v) + (xv\tau^*/2) \sin(\pi/v) + O(\varepsilon), \quad \omega = \pi/v + \varepsilon x \quad (20)$$

Expression (20) holds only locally, and, in view of expression (16) which involves integrations over the infinite frequency domain, must be analytically continued for $-\infty < \omega < +\infty$. To perform this analytic extension the following global continuation of $\cosh \mu$ is performed, valid for positive as well as negative frequencies:

$$\cosh \mu^{(HF)} = -\cos(\pi/v) - \sin(\pi/v) \frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} \quad (21)$$

It is easy to show that for $\tau = \tau^*/\varepsilon$ and $\omega = \pi/v + \varepsilon x$ expressions (20) and (21) agree to leading order. Relation (21) provides the global high-frequency approximation for the propagation constant μ , and is analogous to the low-frequency approximation $\mu^{(LE)}$ derived in the previous section. Employing (21) the high-frequency approximation for $\sinh \mu$ is computed as follows:

$$\sinh \mu^{(HF)} = - \left[\left(\cos(\pi/v) + \sin(\pi/v) \frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} \right)^2 - 1 \right]^{1/2} \quad (1AZ)$$

$$\sinh \mu^{(HF)} = \pm j \left[1 - \left(\cos(\pi/v) + \sin(\pi/v) \frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} \right)^2 \right]^{1/2} \quad (2PZ)$$

$$\sinh \mu^{(HF)} = \left[\left(\cos(\pi/v) + \sin(\pi/v) \frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} \right)^2 - 1 \right]^{1/2} \quad (2AZ) \quad (22)$$

where the $(-)$ and $(+)$ signs in the second of the above expressions hold for $\omega \geq 0$ and $\omega < 0$, respectively. Note, that since the analysis deals with frequency components of the transfer function close to 2PZ, only this PZ and its neighboring AZs are taken into account; higher or lower PZs and AZs are not considered. The sign convention employed in (22) is compatible with that used for deriving the low-frequency approximations for $\sinh \mu$, expressions (8). In addition, it can be easily shown that as $\omega \rightarrow \pm\infty$, the last of the above relations diverges:

$$\sinh \mu^{(HF)} \sim \omega^2 \frac{v^2 \tau}{4\pi} \sin(\pi/v) - \left(\cos(\pi/v) + \frac{\pi \tau}{4} \sin(\pi/v) \right) + O(1/\omega^2), \quad \omega \rightarrow \pm \infty \quad (23)$$

The high-frequency correction to the transfer function, $R^{(HF)}(\omega; p)$, in (19) is now analyzed. Considering the general expression for $R(\omega; p)$, eqn (7), and performing an asymptotic expansion in an $O(\varepsilon)$ neighborhood of 2PZ (by setting $\omega = \pi/v + \varepsilon x$), one obtains the following local high-frequency approximation:

$$R(x; p) = \cos(p\pi/v) + \left[-(xv\tau^*/2) + \frac{\sinh \mu}{\sin(\pi/v)} \right] \sin(p\pi/v) + O(\varepsilon), \quad \omega = \pi/v + \varepsilon x \quad (24)$$

Following the previously outlined procedure, the local expression (24) is extended to the entire frequency domain by performing the substitutions,

$$-(xv\tau^*/2) \rightarrow \frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)}, \quad \sinh \mu \rightarrow \sinh \mu^{(HF)} \quad (25)$$

leading to the following global extension for $R(x; p)$:

$$\hat{R}^{(HF)}(\omega; p) = \cos(p\pi/v) + \left[\frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} + \frac{\sinh \mu^{(HF)}}{\sin(\pi/v)} \right] \sin(p\pi/v) \quad (26a)$$

where depending on the frequency, $\sinh \mu^{(HF)}$ is computed by relations (22). Considering the limiting behavior of $\hat{R}^{(HF)}(\omega; p)$ for large frequencies, one finds that,

$$\lim_{\omega \rightarrow \pm \infty} \hat{R}^{(HF)}(\omega; p) = \cos(p\pi/v) - \cot(\pi/v) \sin(p\pi/v) \quad (26b)$$

In obtaining this limit, relation (23) was taken into account. Clearly, a valid high-frequency correction to the transfer function must decay to zero as $\omega \rightarrow \pm \infty$, hence, it is necessary to subtract from (26a) the residual term $\text{Res}(p) = [\cos(p\pi/v) - \cot(\pi/v) \sin(p\pi/v)]$, leading to the following final expression for $R^{(HF)}(\omega; p)$:

$$\begin{aligned} R^{(HF)}(\omega; p) &= \cos(p\pi/v) + \left[\frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} + \frac{\sinh \mu^{(HF)}}{\sin(\pi/v)} \right] \sin(p\pi/v) - \text{Res}(p) \\ &= \left[\frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} + \frac{\sinh \mu^{(HF)}}{\sin(\pi/v)} + \cot(\pi/v) \right] \sin(p\pi/v) \end{aligned} \quad (26c)$$

At the limit of large positive or negative frequencies the above expression decays to zero, and provides a valid asymptotic approximation to the high-frequency corrections of the transfer function. Taking into account the global expressions of $\sinh \mu^{(HF)}$ in 1AZ, 2PZ and 2AZ (eqn (22)), one obtains the following expressions for $R^{(HF)}(\omega; p)$ in these frequency subdomains:

$$\begin{aligned} R^{(HF)}(\omega; p) &= \sin(p\pi/v) \left[\frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} \right. \\ &\quad \left. - \left[\left(\cot(\pi/v) + \frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} \right)^2 - \frac{1}{\sin^2(\pi/v)} \right]^{1/2} + \cot(\pi/v) \right] \end{aligned} \quad (1AZ)$$

$$R^{(HF)}(\omega; p) = \sin(p\pi/v) \left[\frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} \pm j \left[\frac{1}{\sin^2(\pi/v)} - \left(\cot(\pi/v) + \frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} \right)^2 \right]^{1/2} + \cot(\pi/v) \right] \quad (1PZ)$$

$$R^{(HF)}(\omega; p) = \sin(p\pi/v) \left[\frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} + \left[\left(\cot(\pi/v) + \frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} \right)^2 - \frac{1}{\sin^2(\pi/v)} \right]^{1/2} + \cot(\pi/v) \right] \quad (2AZ) \quad (27)$$

where $(-)$ and $(+)$ signs in the second on the above expressions hold for positive and negative frequencies, respectively; note that relations (27) are valid for both positive and negative frequencies. Substituting (27) into (19), one obtains the following analytic approximation for the high-frequency convolution kernel $r^{(HF)}(t; p)$:

$$r^{(HF)}(t; p) = (2\pi)^{-1} (c_A/l_A) \left\{ \int_{-\infty}^{-\omega_0^{(-)}} R^{(HF)}(\omega; p) \exp(j(c_A/l_A)\omega t) d\omega + \int_{-\omega_0^{(-)}}^{-\omega_0^{(+)}} R^{(HF)}(\omega; p) \exp(j(c_A/l_A)\omega t) d\omega + \int_{\omega_0^{(+)}}^{+\infty} R^{(HF)}(\omega; p) \exp(j(c_A/l_A)\omega t) d\omega \right\} \quad (28)$$

where the appropriate expression for $R^{(HF)}(\omega; p)$ in each integral is determined by relations (27). The quantities $\omega_0^{(\pm)}$ in the limits of the above integrals correspond the bounding frequencies of the second PZ, and are computed by the relation $\cosh \mu^{(HF)} = \pm 1$, i.e., by employing the global approximation for the propagation constant, eqn (21):

$$\omega_0^{(\pm)} = \left[(\pi/v)^2 - 4(\pi/v) \frac{\pm 1 - \cos(\pi/v)}{\tau v \sin(\pi/v)} \right]^{1/2} \quad (29)$$

It can be shown that for $\tau = O(1/\varepsilon)$, $\omega_0^{(\pm)} = \omega_0^{(2)\pm} + O(\varepsilon^2)$, where $\omega_0^{(2)\pm}$ are the asymptotic expressions for the bounding frequencies of 2PZ in the limit of large τ (cf. eqn 6b).

Performing algebraic manipulations, relation (28) can be reduced to the following simplified form:

$$r^{(HF)}(t; p) = (2\pi)^{-1} (c_A/l_A) \sin(p\pi/v) \int_{-\infty}^{+\infty} \left[\cot(\pi/v) + \frac{v\tau}{4} \frac{(\pi/v)^2 - \omega^2}{(\pi/v)} + \frac{v^2\tau}{4\pi} (\omega^2 - \omega_0^{(+2)})^{1/2} (\omega^2 - \omega_0^{(-2)})^{1/2} \right] \exp(-j(c_A/l_A)\omega t) d\omega \quad (30)$$

In obtaining (30), the multi-valued function $f(z) = (z^2 - \omega_0^{(+2)})^{1/2} (z^2 - \omega_0^{(-2)})^{1/2}$ was appropriately defined on the real axis of the complex plane. Introducing the change of variable, $u = (v^2\tau/4\pi)^{1/2} \sin^{1/2}(\pi/v)\omega$, the integral in (30) is expressed as follows:

$$r^{(HF)}(t; p) = (\Omega_0/\pi) \sin(p\pi/v) \sin(\pi/v) \int_{-\infty}^{+\infty} [1 - 2(u^2 - b^2) + 2(u^2 - b^2)^{1/2}(u^2 - 1 - b^2)^{1/2}] \exp(-2j\Omega_0 ut) du \quad (31)$$

where $\Omega_0 = (1/2)(c_A/l_A)(8\pi/v^2\tau)^{1/2}[\sin(\pi/v)]^{-1/2}$, and $b^2 = (\pi/8) \sin(\pi/v) + (1/2)[\cos(\pi/v) - 1]$. The improper integral in (31) was evaluated in closed form by Wang and Lee (1973) using contour integration in the complex plane:

$$\begin{aligned} & \int_{-\infty}^{+\infty} [1 - 2(u^2 - b^2) + 2(u^2 - b^2)^{1/2}(u^2 - 1 - b^2)^{1/2}] \exp(-2j\Omega_0 ut) du \\ &= (\pi/2) \int_0^{2\Omega_0 t} J_0(\sqrt{(2b\Omega_0 t)^2 - (b\tau)^2}) [J_0(\tau) - J_4(\tau)] d\tau \\ &= \frac{2\pi}{\Omega_0 t} J_2(2\Omega_0 t) + 4\pi b \int_0^1 \frac{J_2(2\Omega_0 tv) J_2'(2b\Omega_0 t \sqrt{1-v^2})}{\sqrt{1-v^2}} dv \end{aligned} \quad (32)$$

where $J_k(v)$ denotes the k -th order Bessel function of the first kind. In writing the last of relations (32) certain properties of Bessel functions were taken into account. Combining (31) and (32) one obtains the following closed-form expression for the high-frequency convolution kernel:

$$r^{(HF)}(t; p) = (\Omega_0/2) \sin(p\pi/v) \sin(\pi/v) \int_0^{2\Omega_0 t} J_0(\sqrt{(2b\Omega_0 t)^2 - (b\tau)^2}) [J_0(\tau) - J_4(\tau)] d\tau \quad (33)$$

where only the first of expressions (32) was employed. The improved analytic approximation for the dispersed stress wave in the first layer is provided by expression (18), and assumes the form:

$$\begin{aligned} f(t; p) &\approx f^{(LF)}(t; p) + f^{(HF)}(t; p) = \\ &= f^{(LF)}(t; p) + \int_{-\infty}^{+\infty} \left[(\Omega_0/2) \sin(p\pi/v) \sin(\pi/v) \right. \\ &\quad \left. \times \int_0^{2\Omega_0(t-u)} J_0(\sqrt{[2b\Omega_0(t-u)]^2 - (b\tau)^2}) [J_0(\tau) - J_4(\tau)] d\tau \right] f_L(u) du \end{aligned} \quad (34)$$

The high-frequency oscillatory components of $f^{(HF)}(t; p)$ are of smaller magnitude than those of $f^{(LF)}(t; p)$. It is noted that by construction the analytical solution (34) one takes into account only frequency components of the transfer function close to 1PZ and 2PZ of system II (cf. Fig. (2b)). Components in the vicinities of higher PZs can be dealt with similarly, although their contributions to $f(t; p)$ become smaller as the frequency of the PZ considered increases. This explains the lack of high-frequency oscillations in the numerically computed wave profiles for system I (Fig. 3a).

5. CONCLUDING REMARKS

The dispersion of transient stress waves in a semi-infinite system composed of bi-layered periodic sets was analyzed. The dynamics of this system depends on two non-dimensional structural parameters, namely, τ and v . For large τ and finite v , the system is weakly coupled and possesses small PZs; moreover, the first PZ is degenerate, and possesses

larger width than higher non-degenerate ones. The analysis was based on an approximate inversion of the transfer function of the internal stress, taking into account only frequency components in vicinities of low PZs.

It was shown that, for large values of τ , an inversion taking into account frequencies in the vicinity of the first PZ captures the main characteristics of the dispersed wave. The low-frequency analytic approximation of the transient stress was computed as a superposition of a non-oscillating term proportional to the applied force, and an oscillatory convolution integral. Depending on the values of parameters τ and ν , the oscillatory kernel of the convolution was found to vary in magnitude and frequency, causing overshoot or undershoot of the dispersed wave. The analytical results were in agreement with numerical simulations. High-frequency modulations in the dispersed wave were analytically studied by considering higher-frequency components of the transfer function. The derived high-frequency analytic correction has the form of a double integral and its terms possess small amplitudes. Although such high-frequency modulations are interesting from an analytical point of view, for large τ they possess small amplitudes, and, hence, are of limited practical importance. In such a case it is the low-frequency approximation which mainly determines the dispersed wave.

The analytical formulae developed herein can be of significant practical use when designing layered, composite systems such as stress attenuators or shock isolators. As shown in this and other works in the literature, depending on the elastic and geometric properties of the layers, significant dispersion and undershoot/overshoot of the transient stress wave in the leading layer of the system results. In particular, the observed undershoot of the primary stress pulse in the first layer of system II is attributed to the complicated pattern of reflections and transmissions of the various frequency components of the stress wave at the various layer interfaces of the semi-infinite system; the stress wave undergoes further dispersion as it propagates through the rest of the layered medium.

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